Application of Homotopy Perturbation Method with Chebyshev Polynomials to Nonlinear Problems

Changbum Chun

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea

Reprint requests to C. C.; Fax: +82-31-290-7033; E-mail: cbchun@skku.edu

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In this paper, we present an efficient modification of the homotopy perturbation method by using Chebyshev's polynomials and He's polynomials to solve some nonlinear differential equations. Some illustrative examples are given to demonstrate the efficiency and reliability of the modified homotopy perturbation method.

Key words: Homotopy Perturbation Method; Chebyshev Polynomials; He's Polynomials; Nonlinear Differential Equations.

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1. Introduction

Nonlinear differential equations arise in a wide variety of problems such as fluid dynamics, quantum field theory, and plasma physics to describe the various phenomena. These problems, a limited number of them apart, do not have a precise analytical solution, so these nonlinear equations should be solved using approximate methods.

The homotopy perturbation method (HPM), first proposed by He in 1998, was developed and improved by He [1 – 3]. Very recently, the new interpretation and new development of the homotopy perturbation method have been given and well addressed in [4 – 7]. Homotopy perturbation method [1 – 7] is a novel and effective method, and can solve various nonlinear equations. This method has successfully been applied to solve many types of linear and nonlinear problems, for example, nonlinear oscillators with discontinuities [8], nonlinear wave equations [9], limit cycle and bifurcations [10 – 13], nonlinear boundary value problems [14], asymptotology [15], Volterra's integro-differential equation by El-Shahed [16], some fluid problems [17 – 18], Chen system and other systems of equations [19-20], singular problems [21], and many other problems [22 – 23] and the references therein. These applications also verified that the HPM offers certain advantages over other conventional numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy, and requires large computer power and time. The HPM is better since it does not involve discretization of the variables, hence is free from rounding off errors and does not require large computer memory or time. Recently, some modifications of the Adomian decomposition method by using orthogonal polynomials were obtained by Hosseini [24] and Tien and Chen [25].

In this paper, an efficient modification of the homotopy perturbation method is used to solve some initial differential equations for which an approximation may be required to deal with the source term. In order to make the HPM more effective, He's polynomials and Chebyshev's polynomials are used in the modified homotopy perturbation method. It should be mentioned that the idea of He's polynomials was first suggested by Ghorbani to deal with nonlinear terms when using the HPM [26 – 27], and was then also used in the variational iteration method to deal with nonlinear terms in the correction functional [28]. In this paper, several illustrative examples are given to reveal the efficiency and reliability of the modified homotopy perturbation method.

2. He's Homotopy Perturbation Method

To illustrate the homotopy perturbation method (HPM) for solving nonlinear differential equations, He [1 – 7] considered the following nonlinear differential

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equation:

$$L(u)+R(u)+N(u)=g(x), \quad x\in\Omega, \tag{1}$$

subject to the boundary condition

$$B\left(u\frac{\partial u}{\partial n}\right)=0, \quad x\in\Gamma, \tag{2}$$

where L is a linear operator of highest order, R a linear operator of the remaining linear terms, N a nonlinear operator, B a boundary operator, $g(x)$ a known analytic function, Γ is the boundary of the domain Ω, and $\frac{\partial}{\partial n}$ denotes the differentiation along the normal vector drawn outwards from Ω. He [1–2] constructed a homotopy $v(r,p):\Omega\times[0,1]\to\Re$ which satisfies

$$\begin{aligned}H(v,p)=(1-p)&[L(v)-L(u_0)]\\&+p[L(v)+R(v)+N(v)-g(x)]=0,\end{aligned} \tag{3}$$

which is equivalent to

$$\begin{aligned}H(v,p)=L(v)&-L(u_0)+pL(u_0)\\&+p[R(v)+N(v)-g(x)]=0,\end{aligned} \tag{4}$$

where $p\in[0,1]$ is an embedding parameter, and u_0 is an initial approximation of (3). Obviously, we have:

$$\begin{aligned}H(v,0)&=L(v)-L(u_0)=0,\\H(v,1)&=L(v)+R(v)+N(v)-g(x)=0.\end{aligned} \tag{5}$$

The changing process of p from zero to unity is just that of $H(v,p)$ from $L(v)-L(u_0)$ to $L(v)+R(v)+N(v)-g(x)$. In topology, this is called deformation and $L(v)-L(u_0)$ and $L(v)+R(v)+N(v)-g(x)$ are called homotopic. According to the homotopy perturbation method, the parameter p is used as a small parameter, and the solution of (3) can be expressed as a series in p in the form

$$v=v_0+pv_1+p^2v_2+p^3v_3+\cdots. \tag{6}$$

When $p\to 1$, (4) corresponds to the original one, (1), and (6) becomes the approximate solution of (1), i. e.

$$u=\lim_{p\to 1}v=v_0+v_1+v_2+v_3+\cdots. \tag{7}$$

If (1) admits an unique solution, then this method produces the unique solution. If (1) does not possess an unique solution, the HPM will give a solution among many other possible solutions. The convergence of the series in (7) is discussed by He in [1–2].

3. The Chebyshev-Based Homotopy Perturbation Method

To perform the homotopy perturbation method in such situations as that $g(x)$ is not easily integrable, in general, for an arbitrary natural number m, $g(x)$ may be expressed in the Taylor series,

$$g(x)\approx g_{\mathrm{T},m}=\sum_{i=0}^{m}g_i(x). \tag{8}$$

In this paper, we suggest that $g(x)$ can be expressed in the Chebyshev series,

$$g(x)\approx g_{\mathrm{C},m}(x)=\sum_{i=0}^{m}a_iT_i(x), \tag{9}$$

where the $T_i(x)$ are the Chebyshev polynomials of the first kind,

$$\begin{aligned}&T_0(x)=1, \quad T_1(x)=x,\\&T_2(x)=2x^2-1, \quad T_3(x)=4x^3-3x,\end{aligned} \tag{10}$$

and, in general, the following recursive relation are satisfied:

$$T_{k+1}(x)=2xT_k(x)-T_{k-1}(x), \quad k\geq 1. \tag{11}$$

We recall that the Chebyshev polynomials of the first kind are orthogonal defined on the interval $[-1,1]$ with respect to the weight function $w(x)=(1-x^2)^{-1/2}$. In order to solve an initial value problem in a large enough region, for example, on a general interval [a, b], these polynomials can be extended to the interval [a, b] by using the change of variables $\tilde{x}=\frac{1}{2}[(b-a)x+a+b]$ to transform the numbers in the interval [−1, 1] into the corresponding numbers in the interval [a, b].

To deal with the nonlinear term $N(v)$, we will employ He's polynomials, which were first considered in [26–27], defined by

$$N(v_0,v_1,\cdots,v_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{n}p^kv_k\right)_{p=0}, \quad n=0,1,\ldots, \tag{12}$$

and satisfy the relation

$$\begin{aligned}N(v)=N(v_0)&+N(v_0,v_1)p+N(v_0,v_1,v_2)p^2\\&+\cdots+N(v_0,v_1,\cdots,v_n)p^n+\cdots.\end{aligned} \tag{13}$$

Substituting (6), (9), and (13) into (4), and equating coefficients of like powers of p, we obtain

$$p^0: \quad L(v_0)-L(u_0)=0, \tag{14}$$

$$p^1: \quad L(v_1)+L(u_0)+R(v_0)+N(v_0) - g_{\mathrm{C},m}(x)=0, \tag{15}$$

$$p^2: \quad L(v_2)+R(v_1)+N(v_0,v_1)=0, \tag{16}$$

$$p^3: \quad L(v_3)+R(v_2)+N(v_0,v_1,v_2)=0, \tag{17}$$

$$\vdots$$

$$p^{n+1}: \quad L(v_{n+1})+R(v_n) +N(v_0,v_1,\cdots,v_n)=0, \tag{18}$$

and so on, where $g_{\mathrm{C},m}(x)$ is defined as in (9). The successive components of the solution can be completely determined, and the solution is thus obtained. For concrete problems, the M-term partial sum

$$\Phi_M=\sum_{k=0}^{M-1} v_k \tag{19}$$

may be used to give the approximate solution.

The Chebyshev-based HPM introduced above will be illustrated by discussing the following examples.

4. Numerical Examples

The Chebyshev-based method will be used to solve the two initial ordinary differential equations with the help of Maple package.

Example 1. We consider for $0 \le x \le 1$, [24]

$$u''+xu'+x^2u^3=(2+6x^2)\mathrm{e}^{x^2}+x^2\mathrm{e}^{3x^2}, \tag{20}$$

$$u(0)=1, \quad u'(0)=0, \tag{21}$$

with the exact solution $u(x)=\mathrm{e}^{x^2}$. In an operator form, (20) can be written as

$$L(u)+R(u)+N(u)=g(x), \tag{22}$$

where $L=\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $R=x\frac{\mathrm{d}}{\mathrm{d}x}$, $N(u)=x^2u^3$, and $g(x)=(2+6x^2)\mathrm{e}^{x^2}+x^2\mathrm{e}^{3x^2}$.

According to (12), He's polynomials are found to be

$$\begin{aligned}
&N(v_0)=x^2v_0^3,\\
&N(v_0,v_1)=x^2(3v_0^2v_1),\\
&N(v_0,v_1,v_2)=x^2(3v_0^2v_2+3v_0v_1^2),\\
&N(v_0,v_1,v_2,v_3)=x^2(3v_0^2v_3+6v_0v_1v_2+v_1^3),\\
&\vdots
\end{aligned} \tag{23}$$

We note that the techniques such as the Adomian decomposition method are not easy to apply if $g(x)$ is not easily integrable. However, this difficulty can be overcome by considering the Taylor series or the Chebyshev series of $g(x)$.

Now we let $m=6$, and $M=7$. A Taylor polynomial of $g(x)$ is

$$g_{\mathrm{T},6}(x)=2+9x^2+10x^4+\frac{47}{6}x^6. \tag{24}$$

In view of the homotopy (4), we construct the following homotopy:

$$v''+p[xv'+x^2v^3-g_{\mathrm{T},6}(x)]=0. \tag{25}$$

Substituting (6) and using (13) with He's polynomials given by (23) into the homotopy (25) and equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$p^0: \quad v_0''=0, \quad v_0'(0)=0, \quad v_0(0)=1, \tag{26}$$

$$p^1: \quad v_1''+xv_0'+x^2-g_{\mathrm{T},6}(x)=0, \quad v_1'(0)=0, \quad v_1(0)=0, \tag{27}$$

$$p^2: \quad v_2''+xv_1'+3x^2v_1=0, \quad v_2'(0)=0, \quad v_2(0)=0, \tag{28}$$

$$p^3: \quad v_3''+xv_2'+3x^2(v_2+v_1^2)=0, \quad v_3'(0)=0, \quad v_3(0)=0, \tag{29}$$

$$p^4: \quad v_4''+xv_3'+x^2(3v_3+6v_1v_2+v_1^3)=0, \quad v_4'(0)=0, \quad v_4(0)=0, \tag{30}$$

$$\vdots$$

Consequently, solving the above equations by the help of Maple, we obtain

$$\begin{aligned}
u_{\mathrm{T}}(x)&=\sum_{i=0}^{6} v_i=1+2.01582x^2-0.48273x^3\\
&+4.53721x^4-11.27641x^5+18.20236x^6\\
&-13.46358x^7+\cdots+1.07441x^{38}\\
&-0.16931x^{39}+0.01288x^{40}.
\end{aligned} \tag{31}$$

The absolute error of $u_{\mathrm{T}}(x)$ is presented in Figure 1.

Now, we use the Chebyshev expansion for $g(x)$, in this case we have

$$g_{\mathrm{C},6}(x)=\sum_{i=0}^{6} a_iT_i(2x-1), \quad 0\le x\le 1, \tag{32}$$

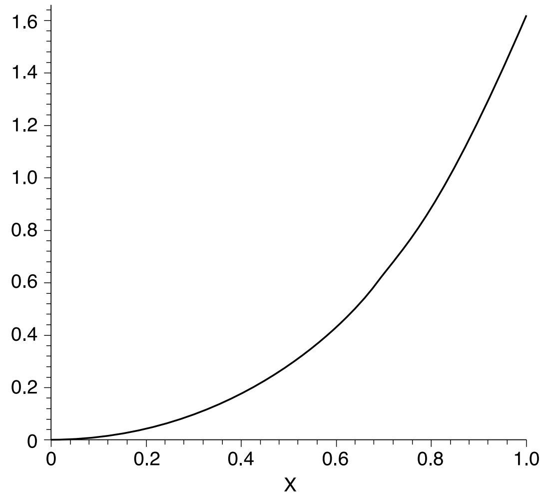

Fig. 1. Absolute error $|u_{\mathrm{T}}(x) - u(x)|$ for Example 1.

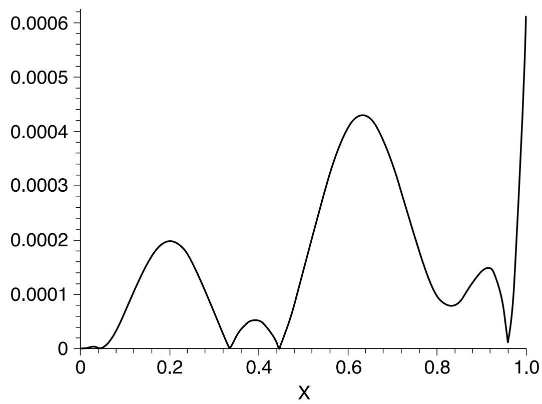

Fig. 2. Absolute error $|u_{\mathrm{C}}(x) - u(x)|$ for Example 1.

where

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{g(0.5x+0.5)T_0(x)}{\sqrt{1-x^2}}\mathrm{d}x, \tag{33}$$

$$a_i = \frac{2}{\pi}\int_{-1}^{1} \frac{g(0.5x+0.5)T_i(x)}{\sqrt{1-x^2}}\mathrm{d}x, \quad i = 1,2,\ldots, \tag{34}$$

(32) then gives

$$\begin{aligned} g_{\mathrm{C},6}(x) = {} & 2.03164 - 2.89636x + 51.47813x^2 \\ & - 226.97633x^3 + 560.26700x^4 \\ & - 623.30055x^5 + 281.17276x^6. \end{aligned} \tag{35}$$

We construct the following homotopy:

$$v'' + p[xv' + x^2v^3 - g_{\mathrm{C},6}(x)] = 0. \tag{36}$$

Substituting (6) and using (13) with He's polynomials given by (23) into the homotopy (36) and equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$p^0: \quad v_0'' = 0, \quad v_0'(0) = 0, \quad v_0(0) = 1, \tag{37}$$

$$\begin{aligned} p^1: \quad & v_1'' + xv_0' + x^2 - g_{\mathrm{C},6}(x) = 0, \\ & v_1'(0) = 0, \quad v_1(0) = 0, \end{aligned} \tag{38}$$

$$\begin{aligned} p^2: \quad & v_2'' + xv_1' + 3x^2v_1 = 0, \\ & v_2'(0) = 0, \quad v_2(0) = 0, \end{aligned} \tag{39}$$

$$\begin{aligned} p^3: \quad & v_3'' + xv_2' + 3x^2(v_2 + v_1^2) = 0, \\ & v_3'(0) = 0, \quad v_3(0) = 0, \end{aligned} \tag{40}$$

$$\begin{aligned} p^4: \quad & v_4'' + xv_3' + x^2(3v_3 + 6v_1v_2 + v_1^3) = 0, \\ & v_4'(0) = 0, \quad v_4(0) = 0, \end{aligned} \tag{41}$$

$$\vdots$$

Consequently, solving the above equations by the help of Maple, we obtain

$$\begin{aligned} u_{\mathrm{C}}(x) = \sum_{i=0}^{6} v_i = {} & 1 + 1.02582x^2 - 0.48273x^3 \\ & + 4.03721x^4 - 11.27641x^5 + 18.03569x^6 \\ & - 13.46358x^7 + \cdots + 1.07441x^{38} \\ & - 0.16931x^{39} + 0.01288x^{40}. \end{aligned} \tag{42}$$

We present the absolute error of $u_{\mathrm{C}}(x)$ in Figure 2.

From Figure 1 and 2, it is observed that the Chebyshev-based homotopy perturbation method (36) approximates the solution more accurately and efficiently than the Taylor-based method (25).

Example 2. We consider for $0 \le x \le 1$, [24]

$$u'' + uu' = x\sin(2x^2) - 4x^2\sin(x^2) + 2\cos(x^2), \tag{43}$$

$$u(0) = 0, \quad u'(0) = 0, \tag{44}$$

with the exact solution $u(x) = \sin(x^2)$. In an operator form, (43) can be written as

$$L(u) + N(u) = g(x), \tag{45}$$

where $L = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $N(u) = uu'$, and $g(x) = x\sin(2x^2) - 4x^2\sin(x^2) + 2\cos(x^2)$.

He's polynomials for the nonlinear term $N(v) = vv'$ in this case are given by

$$\begin{aligned} &N(v_0) = v_0 v_0', \\ &N(v_0, v_1) = v_1 v_0' + v_0 v_1', \\ &N(v_0, v_1, v_2) = v_2 v_0' + v_1 v_1' + v_0 v_2', \\ &N(v_0, v_1, v_2, v_3) = v_3 v_0' + v_2 v_1' + v_1 v_2' + v_0 v_3', \\ &\quad \vdots \end{aligned} \tag{46}$$

Now we let $m = 10$, and $M = 11$. A Taylor polynomial of $g(x)$ is given by

$$g(x) = 2 + 2x^3 - 5x^4 - \frac{4}{3}x^7 + \frac{3}{4}x^8 + O(x^{11}), \tag{47}$$

so that

$$g_{\mathrm{T},10}(x) = 2 + 2x^3 - 5x^4 - \frac{4}{3}x^7 + \frac{3}{4}x^8. \tag{48}$$

In view of the homotopy (4) and the initial conditions (44), we construct the following homotopy:

$$v'' - 2 + p[2 + vv' - g_{\mathrm{T},10}(x)] = 0. \tag{49}$$

Substituting (6) and using (13) with He's polynomials given by (46) into the homotopy (49) and equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$\begin{aligned} p^0: \quad & v_0'' - 2 = 0, \\ & v_0'(0) = 0, \quad v_0(0) = 0, \end{aligned} \tag{50}$$

$$\begin{aligned} p^1: \quad & v_1'' + 2 + v_0 v_0' - g_{\mathrm{T},10}(x) = 0, \\ & v_1'(0) = 0, \quad v_1(0) = 0, \end{aligned} \tag{51}$$

$$\begin{aligned} p^2: \quad & v_2'' + v_1 v_0' + v_0 v_1 = 0, \\ & v_2'(0) = 0, \quad v_2(0) = 0, \end{aligned} \tag{52}$$

$$\begin{aligned} p^3: \quad & v_3'' + v_2 v_0' + v_1 v_1' + v_0 v_2' = 0, \\ & v_3'(0) = 0, \quad v_3(0) = 0, \end{aligned} \tag{53}$$

$$\vdots$$

Consequently, solving the above equations by the help of Maple, we obtain

$$\begin{aligned} u_{\mathrm{T}}(x) &= \sum_{i=0}^{10} v_i = 0.99999x^2 + 0.00003x^3 \\ &- 0.33385x^4 + 0.00472x^5 - 0.47974x^6 + \cdots \\ &+ 0.24791 \cdot 10^{-13} x^{65} - 0.16651 \cdot 10^{-14} x^{66} \\ &+ 0.52738 \cdot 10^{-16} x^{67}. \end{aligned} \tag{54}$$

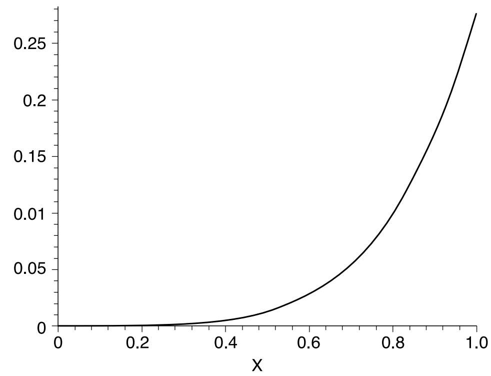

Fig. 3. Absolute error $|u_{\mathrm{T}}(x) - u(x)|$ for Example 2.

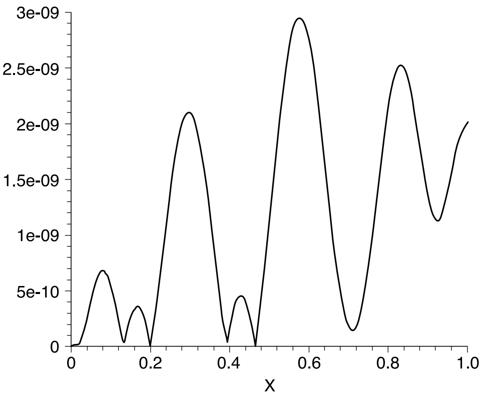

Fig. 4. Absolute error $|u_{\mathrm{C}}(x) - u(x)|$ for Example 2.

The absolute error of $u_{\mathrm{T}}(x)$ is presented in Figure 3.

Now, we use the Chebyshev expansion for $g(x)$, in this case we have

$$g_{\mathrm{C},10}(x) = \sum_{i=0}^{10} a_i T_i(2x-1), \quad 0 \le x \le 1, \tag{55}$$

where the a_0 and a_i, $i = 1, 2, \ldots$, are defined as in (33) and (34), respectively. After the use of Maple, (55) yields

$$\begin{aligned} g_{\mathrm{C},10}(x) &= 1.99999 + 0.00016x - 0.00625x^2 \\ &+ 2.09443x^3 - 5.72538x^4 + 3.20979x^5 \\ &- 8.68549x^6 + 13.29048x^7 - 14.17972x^8 \\ &+ 8.33614x^9 - 1.71016x^{10}. \end{aligned} \tag{56}$$

In view of the homotopy (4) and the initial conditions (44), we construct the following homotopy:

$$v'' - 2 + p[2 + vv' - g_{\mathrm{C},10}] = 0. \tag{57}$$

Substituting (6) and using (13) with He's polynomials given by (46) into the homotopy (57) and equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$p^0: \quad v_0'' - 2 = 0, \quad v_0'(0) = 0, \quad v_0(0) = 0, \tag{58}$$

$$p^1: \quad v_1'' + 2 + v_0 v_0' - g_{C,10}(x) = 0, \quad v_1'(0) = 0, \quad v_1(0) = 0, \tag{59}$$

$$p^2: \quad v_2'' + v_1 v_0' + v_0 v_1 = 0, \quad v_2'(0) = 0, \quad v_2(0) = 0, \tag{60}$$

$$p^3: \quad v_3'' + v_2 v_0' + v_1 v_1' + v_0 v_2' = 0, \quad v_3'(0) = 0, \quad v_3(0) = 0, \tag{61}$$

$$\vdots$$

Consequently, solving the above equations by the help of Maple, we obtain

$$u_C(x) = \sum_{i=0}^{10} v_i = 0.99999x^2 + 0.00005x^3 - 0.33437x^4 + 0.00944x^5 - 0.67059x^6 + \cdots + 0.49582 \cdot 10^{-13} x^{65} - 0.33301 \cdot 10^{-14} x^{66} + 0.10548 \cdot 10^{-15} x^{67}. \tag{62}$$

We present the absolute error of $u_C(x)$ in Figure 4.

From Figure 3 and 4, it is also observed that the Chebyshev-based method (57) approximates the solution much more accurately and efficiently as more terms are considered in the approximate numerical solution of the problem (43)-(44) than the Taylor-based method (49) in the application of the homotopy perturbation method. This approach can be applied to solve other kinds of nonlinear problems.

5. Conclusion

In this work, we successfully apply He's homotopy perturbation method in combination with Chebyshev's polynomials to solve the differential equations with source term for which the Taylor series is required. We demonstrated that the Chebyshev-based HPM shows a much better performance over the Taylor-based HPM in handling nonlinear differential equations with the not easily integrable source term.

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[1] J. H. He, Comput. Methods Appl. Mech. Eng. **178**, 257 (1999).
[2] J. H. He, Int. J. Nonlinear Mech. **35**, 37 (2000).
[3] J. H. He, Appl. Math. Comput. **135**, 73 (2003).
[4] J. H. He, Int. J. Mod. Phys. B **20**, 2561 (2006).
[5] J. H. He, The full text is available at: http://works.bepress.com/ji_huan_he/37.
[6] J. H. He, Int. J. Mod. Phys. B **20**, 1144 (2006).
[7] J. H. He, The full paper can be downloaded from: http://www.worldscinet.com/cgi-bin/details.cgi?id=jsname:ijmpb&type=current.
[8] J. H. He, Appl. Math. Comput. **151**, 287 (2004).
[9] J. H. He, Chaos, Solitons, and Fractals **26**, 695 (2005).
[10] J. H. He, Int. J. Nonlinear Sci. Numer. Simul. **6**, 207 (2005).
[11] J. H. He, Phys. Lett. A **347**, 228 (2005).
[12] J. H. He, Chaos, Solitons, and Fractals **26**, 827 (2005).
[13] J. H. He, Phys. Rev. Lett. **90** Art. No. 174301 (2003).
[14] J. H. He, Phys. Lett. A **350**, 87 (2006).
[15] J. H. He, Appl. Math. Comput. **156**, 591 (2004).
[16] M. El-Shahed, Int. J. Nonlinear Sci. Numer. Simul. **6**, 163 (2005).
[17] A. M. Siddiqui, R. Mahmood, and Q. K. Ghori, Int. J. Nonlinear Sci. Numer. Simul. **7**, 7 (2006).
[18] A. M. Siddiqui, R. Mahmood, and Q. K. Ghori, Int. J. Nonlinear Sci. Numer. Simul. **7**, 15 (2006).
[19] M. S. H. Chowdhury and I. Hashim, Nonlinear Anal. Ser. B: Real World Appl. **10**, 381 (2009)
[20] M. S. H. Chowdhury and I. Hashim, Nonlinear Anal. Ser. B: Real World Appl. **10**, 104 (2009)
[21] I. Hashim and M. S. H. Chowdhury, Phys. Lett. A **372**, 470 (2008).
[22] L. Cveticanin, Chaos, Solitons, and Fractals **30**, 1221 (2006).
[23] X. C. Cai, W. Y. Wu, and M. S. Li, Int. J. Nonlinear Sci. Numer. Simul. **7**, 109 (2006).
[24] M. M. Hosseini, Appl. Math. Comput. **175**, 1685 (2006).
[25] W.-C. Tien and C.-K. Chen, Chaos, Solitons, and Fractals **39**, 2093 (2009).
[26] A. Ghorbani and J. S. Nadjfi, Int. J. Nonlinear Sci. Numer. Simul. **8**, 229 (2007)
[27] A. Ghorbani, Chaos, Solitons, and Fractals **39**, 1486 (2009).
[28] M. A. Noor and S. T. Mohyud-Din, Int. J. Nonlinear Sci. Numer. Simul. **9**, 141 (2008).